EPFL - Fall Semester 2023-2024	Jérôme Scherer
Cohomology Rings	**Final Exam**
MATHEMATICS	15 January 2024
Duration : 180 minutes	Number of points : 54

Name		First name	
Signature		Sciper	

Justifications and explanations based on the theory seen in class are required in each exercise. Indicate not only that such and such result has been proved in class, but also what this result says (unless it has a clear name like Künneth Formula or universal coefficient Theorem).
The number of points for each part indicates also how detailed the answer should be.

The following table is reserved for correction.

Ex. 1 (30 pts)		Ex. 2 (24 pts)		Bonus (12 pts)	
Written Ex.		Oral Ex.		**Final Grade**	

Question 1. Cohomology of cyclic groups. ($4 + 4 + 2 + 3 + 7 + 2 + 4 + 4 = 30$ points) Let $C = \langle t \mid t^5 \rangle$ be a cyclic group of order 5. The aim of this exercise is to give a detailed computation of the cohomology ring $H^*(C; \mathbb{F}_5)$.

(a) Give a periodic resolution of the trivial $\mathbb{Z}C$-module $\mathbb{Z}$ by free C-modules $F_\bullet \to \mathbb{Z}$. Provide an explicit formula for the augmentation and the differentials $d_n \colon F_n \to F_{n-1}$ for all $n \geq 1$, check it is exact, and identify the kernel of d_1 as a $\mathbb{Z}C$-module.

(b) Identify the cochain complex $\mathrm{Hom}(F_\bullet, \mathbb{F}_5)$ and cocycles representing a set of generators for $H^n(C; \mathbb{F}_5)$ for all $n \geq 0$ (and explain why they are so).

(c) If u is a generator of $H^1(C; \mathbb{F}_5)$ explain why $u^2 = u \cup u = 0$.

(d) Give a formula for the differential of the chain complex $F_\bullet \otimes_\mathbb{Z} F_\bullet$ and define what a diagonal approximation is.

(e) Define a map $\Delta \colon F_\bullet \to F_\bullet \otimes_\mathbb{Z} F_\bullet$ as follows. In degree n we take the sum of $\Delta_{p,q} \colon F_n \to F_p \otimes F_q$ for $p + q = n$, where $\Delta_{p,q}$ is C-equivariant and sends the generator 1 to $1 \otimes 1$ if p is even, $1 \otimes t$ if p is odd and q is even, and

$$\sum_{0 \leq i < j < 5} t^i \otimes t^j \quad \text{if } p \text{ and } q \text{ are odd}$$

Check that Δ is a diagonal approximation by doing the computation up to degree 2 (not involving F_3).

(f) Use Δ to compute $u^2 = u \cup u$ and $v \cup u$ where v is a generator in $H^2(C; \mathbb{F}_5)$.

(g) Explain how to obtain a class $V \in H^2(C; \mathbb{Z})$ inducing a periodicity isomorphism

$$V \cup - \colon H^k(C; \mathbb{F}_5) \to H^{k+2}(C; \mathbb{F}_5)$$

You can refer to all the results we have seen in class and exercises to justify your answer.

(h) Establish an isomorphism of graded $\mathbb{F}_5$-algebras $\mathbb{F}_5[a] \otimes E(b) \to H^*(C; \mathbb{F}_5)$ where a is a polynomial generator in degree 2 and b an exterior generator in degree 1.

Question 2. The James construction. ($2 + 4 + 4 + 2 + 4 + 4 + 2 + 2 = 24$ points + Bonus) The goal of this exercise is to define and analyze the space $J(X)$, which can be thought of as a free topological monoid on X. We will then illustrate this in the case of the 2-sphere and compute $H^*(J(S^2); \mathbb{Z})$ as a graded ring, and even as a graded *Hopf algebra* in the second part of this exercise. We will see it is a *divided polynomial algebra.*

Let X be a pointed space and 1 be its base point. We define $J(X)$ to be the quotient of $\coprod_{k \geq 1} X^k$ by identifying

$$(x_1, x_2, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_k) \in X^k \quad \text{with} \quad (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) \in X^{k-1}$$

for all $k \geq 1$ and all $1 \leq i \leq k$. The image in $J(X)$ of X^k is written $J_k(X)$. Hence $J_1(X)$ is homeomorphic to X and $J_2(X)$ is the quotient of $X \times X$ by the relation $(x, 1) \sim (1, x)$ for all $x \in X$.

Note that $J_2(X)$ contains $J_1(X)$ as the image of $X \times 1$ or $1 \times X$. In general (we admit that) there is a filtration (the so-called James filtration)

$$X = J_1(X) \subset J_2(X) \subset \cdots \subset J_k(X) \subset J_{k+1}(X) \subset \cdots \subset J(X) = \bigcup J_k(X)$$

Part A. The graded algebra structure.

(a) Let X be space. Define a product on $H^*(X;\mathbb{Z}) \otimes_{\mathbb{Z}} H^*(X;\mathbb{Z})$ so that the cohomological cross product $H^*(X;\mathbb{Z}) \otimes_{\mathbb{Z}} H^*(X;\mathbb{Z}) \to H^*(X \times X;\mathbb{Z})$ becomes a homomorphism of algebras.

(b) If X is a connected space with $H_n(X;\mathbb{Z})$ a finitely generated free abelian group for all n, show that the cohomological cross product is an isomorphism.

(c) From now on $X = S^2$. Show that $J_2(S^2)$ is obtained from $S^2 = J_1(S^2)$ by adding a single 4-cell. Explain how the quotient map $S^2 \times S^2 \to J_2(S^2)$ induces an isomorphism on $H_4(-;\mathbb{Z})$. More generally we admit that $J_m(S^2)$ has a CW-structure with exactly one cell in every even dimension $0, 2, \ldots, 2m$ and that the quotient map $(S^2)^m \to J_m(S^2)$ induces an isomorphism on $H_{2m}(-;\mathbb{Z})$.

(d) Compute $H^n(J(S^2);\mathbb{Z})$ for all $n \geq 0$.

(e) Identify $H^*((S^2)^k;\mathbb{Z})$ as a graded ring for $k = 1, 2$. State the result without proof for $k \geq 2$.

(f) Consider the quotient map $q\colon S^2 \times S^2 \to J_2(S^2)$. If $u \in H^2(J_2(S^2);\mathbb{Z})$ is a generator, compute $q^*(u)$ and then the cup product $u^2 = u \cup u \in H^4(J_2(S^2);\mathbb{Z})$.

(g) More generally compute the iterated cup product $u^n \in H^{2n}(J(S^2);\mathbb{Z})$ for all $n \geq 1$.

Hint. It is $n!$ times a generator in degree $2n$.

(h) Prove that there are generators $u_n \in H^{2n}(J(S^n);\mathbb{Z})$ such that $u_n \cup u_m = \binom{n+m}{n} u_{n+m}$.

Part B. Bonus. (4 + 2 + 3 + 1 + 2 = 12 points)

(a) Recall that an H-space is a pointed space with a multiplication $\mu\colon X \times X \to X$ for which the base point acts as a unit up to homotopy. Concatenation $X^k \times X^\ell \to X^{k+\ell}$ induces a map on quotients $J_k(X) \times J_\ell(X) \to J_{k+\ell}(X)$. Show that these maps are compatible with the James filtration and so endow $J(X)$ with an associative H-space structure.

(b) If X is a connected H-space with $H_n(X;\mathbb{Z})$ a finitely generated free abelian group for all n, explain how μ induces an algebra homomorphism $\Delta\colon H^*(X;\mathbb{Z}) \to H^*(X;\mathbb{Z}) \otimes_{\mathbb{Z}} H^*(X;\mathbb{Z})$, called *coproduct* (using the algebra structure from (a) and the result (b) in Part A).

(c) For X as in (b), show that $H^*(X;\mathbb{Z})$ is a graded algebra where $H^0(X;\mathbb{Z}) \cong \mathbb{Z}$ is generated by 1 and such that

$$\Delta(x) = 1 \otimes x + x \otimes 1 + \sum x'_i \otimes x''_{n-i}$$

for any cohomology class x of degree > 0, where x'_i, x''_i are cohomology classes of degree $i > 0$. This amounts basically to saying that it is a *graded Hopf algebra*.

Hint. Compute the image of $\mu^*(x)$ under $(i_1)^*$ and $(i_2)^*$ where $i_1, i_2\colon X \hookrightarrow X \times X$ are the two canonical inclusions.

(d) Show that $u = u_1$ is *primitive* in the graded Hopf algebra $H^*(J(S^2);\mathbb{Z})$, i.e. $\Delta(u) = 1 \otimes u + u \otimes 1$.

(e) Compute the coproduct $\Delta(u^n)$ and prove that $\Delta(u_n) = \sum_{i=0}^{n} u_i \otimes u_{n-i}$ where $u_0 = 1$.